

# Quantum $AdS_5 \times S^5$ superstring in the $AdS$ light-cone gauge

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## Abstract

We consider the  $AdS_5 \times S^5$  superstring in the light-cone gauge adapted to a massless geodesic in  $AdS_5$  in the Poincaré patch. The resulting action has a relatively simple structure which makes it a natural starting point for various perturbative quantum computations. We illustrate the utility of this AdS light-cone gauge action by computing the 1-loop and 2-loop corrections to the null cusp anomalous dimension reproducing in a much simpler and efficient way earlier results obtained in conformal gauge. This leads to a further insight into the structure of the superstring partition function in non-trivial background.

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# 1 Introduction

The superstring theory in  $AdS_5 \times S^5$  [1] has a complicated form and the problem of finding its exact quantum spectrum appears to be a non-trivial one. This is a Green-Schwarz [2] type theory, so a natural way to address the question of its quantization is to use, as in flat space, a light-cone type gauge.

There are two natural choices of light-cone gauge in AdS corresponding to the two inequivalent choices of a massless geodesic: (1) the one running entirely within  $AdS_5$  or (2) the one wrapping a big circle of  $S^5$ . The latter choice corresponds to expanding near the “plane-wave” vacuum [3, 4, 5, 6, 7, 8] and it was widely used in recent studies of the AdS/CFT duality as it is related to the natural “ferromagnetic” (or “magnon”) spin chain vacuum on the gauge theory side (see, e.g., [9, 10] for reviews). The resulting superstring action has a rather involved non-polynomial form and thus is not a simple starting point for “first-principles” quantization.

The former choice of light-cone gauge [11, 12], in which the massless geodesic runs entirely within the (Poincaré patch of)  $AdS_5$ , leads to a simpler action containing terms at most quartic in fermions. However, the importance of the corresponding light-cone vacuum state on the gauge-theory side is not immediately clear; for this reason this light-cone gauge choice received previously less attention (see, however, [13, 14, 15, 16, 17, 18]). In particular, there were practically no studies of the corresponding quantum theory.

The aim of the present paper is to initiate the exploration of the AdS light-cone gauge action at the quantum level. We shall demonstrate that it leads to a consistent definition of the quantum superstring theory by repeating the computations of the 1-loop [6, 19] and 2-loop [20, 21] corrections to the anomaly of the null cusp Wilson line [22] or to the leading term in the large spin expansion of the energy of the folded spinning string in AdS space in global coordinates. We shall reproduce the previous results in a much simpler way, thus providing evidence for the utility of this light-cone gauge.<sup>1</sup> We shall verify, in particular, the cancellation of the 1-loop and 2-loop UV divergences. The cusp anomaly coefficients we will find match the strong-coupling Bethe ansatz [23] predictions [24, 25]; this provides evidence for the quantum integrability of the superstring formulated in the

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<sup>1</sup>The previous computations for the null cusp were carried out in the conformal gauge where, in contrast to the AdS light-cone gauge, the bosonic fluctuations mix in a nontrivial way leading to an off-diagonal propagator and thus complicating the analysis beyond the 1-loop level.

AdS light-cone gauge.<sup>2</sup>

In the companion paper [27] we shall use this approach to evaluate the 2-loop correction to the energy of the folded spinning string with an extra orbital momentum  $J$  in  $S^5$  [6] in the scaling limit when  $\ln S \gg 1$  and  $\frac{J}{\sqrt{\lambda \ln S}} = \text{fixed}$ . Our light-cone gauge result, which is different from the one found using the conformal gauge [28], turns out to be in complete agreement with the Asymptotic Bethe Ansatz calculation of [29] and with the result of [30] which generalizes the connection [31] between the scaling function and the  $O(6)$  sigma model.

A potential future application of the AdS light-cone gauge approach, which motivates our present interest in it, is the study of its near-flat-space expansion aimed at constructing the inverse string tension expansion of the energies of quantum string states with finite quantum numbers (cf. [32]). There are several complications along the way. One of them is the mixing of the “center-of-mass” or superparticle [33, 18] modes with the oscillation string modes (they do not decouple in curved space). Another thorny issue is the realization of the superconformal algebra on excited quantum string states. While representations will be classified by the same quantum numbers as for string in AdS global coordinates, the use of Poincaré coordinates provides new possibilities for constructing the excited string modes and realize the superconformal algebra. Indeed, the AdS light-cone gauge action of [11, 12] is constructed in the Poincaré patch and its Hamiltonian  $P^-$ , whose eigenstates are the quantum string states, is not directly related to the global AdS energy  $E$  (moreover, the two operators do not commute).<sup>3</sup> We hope to return to these issues in the future.

The main part of this paper is organized as follows. In section 2 we shall review the light-cone gauge action of [11, 12] and discuss its simplest “ground state” solution corresponding to the massless geodesic in  $AdS_5$  in Poincaré patch. Expanding near this ground state one finds the same small fluctuation spectrum – 8+8 massless bosonic+fermionic degrees of freedom – as in flat space and the corresponding partition function is trivial.

In section 3 we shall consider a non-trivial solution of the light-cone gauge action representing an open-string euclidean world surface ending on a null cusp on the boundary

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<sup>2</sup>The classical integrability of the  $AdS_5 \times S^5$  superstring [26] on the space of physical degrees of freedom in the AdS light-cone gauge was discussed in [15].

<sup>3</sup>A relation between the descriptions of the quantum particle states based on  $P^-$  and based on  $E$  was discussed in [34, 18].

of  $AdS_5$  [22]. This is still a simple “homogeneous” solution – the coefficients in the light-cone gauge action expanded near it turn out to be constant. As was argued in [19], this world surface is the same (up to an  $SO(2, 4)$  transformation and a euclidean continuation) as the one describing the asymptotic large spin limit [6, 35] of the folded spinning string [5] when the folds approach the boundary. This relation identifies, from a string theory standpoint, the anomaly of a null cusp Wilson line and the large spin limit of the anomalous dimension of twist-2 operators, the former being given by the string partition function in the null cusp background. Expanding the (euclidean analog of the) light-cone gauge action near this null cusp solution we find the same small fluctuation spectrum as in [6, 19] and thus the same 1-loop correction to the cusp anomaly function.

In section 4 we shall extend the computation of the string partition function in the null cusp background to 2-loop order. An important difference compared to the corresponding conformal gauge computation [20, 21] is that one of the massive bosonic fluctuations acquires a nontrivial (and divergent) 1-point function through a tadpole graph with a single fermionic loop. As a result, there are non-vanishing connected but non-1PI contributions to the 2-loop partition function. Summing them together with the 1PI contributions leads to cancellation of all UV divergences and reproduces the Catalan’s constant coefficient in the 2-loop cusp anomaly found earlier [20, 21] by a substantially more involved computation in the conformal gauge.

## 2 Superstring action in the AdS light-cone gauge

Let us begin with a review of the structure of the  $AdS_5 \times S^5$  action in the AdS light-cone gauge [11, 12].

We will use the  $AdS_5 \times S^5$  metric in the Poincaré patch ( $m = 0, 1, 2, 3$ ;  $M = 1, \dots, 6$ )

$$ds^2 = z^{-2}(dx^m dx_m + dz^M dz^M) = z^{-2}(dx^m dx_m + dz^2) + du^M du^M , \quad (2.1)$$

$$x^m x_m = x^+ x^- + x^* x , \quad x^\pm = x^3 \pm x^0 , \quad x = x^1 + ix^2 , \quad (2.2)$$

$$z^M = zu^M , \quad u^M u^M = 1 , \quad z = (z^M z^M)^{1/2} \equiv e^\phi . \quad (2.3)$$

As discussed in [11, 12], starting with the action of [1] in the above coordinates and fixing the  $\kappa$ -symmetry light-cone gauge  $\Gamma^+ \vartheta^I = 0$  on the two 10-d Majorana-Weyl GS spinors  $\vartheta^I$ , one may also choose the following analog of the conformal gauge

$$\sqrt{-g} g^{\alpha\beta} = \text{diag}(-z^2, z^{-2}) . \quad (2.4)$$

Since the resulting action contains  $x^-$  only in the  $\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^+\partial_\beta x^-$  term it admits a simple solution

$$x^+ = p^+ \tau , \quad (2.5)$$

which thus can be consistently imposed as a constraint additional to (2.4) to completely fix the two-dimensional diffeomorphism invariance. With this choice  $x^-$  decouples from the action (it may be determined from the equations of motion for  $g_{\alpha\beta}$  or the analog of the Virasoro constraints where it appears only linearly).<sup>4</sup>

The resulting AdS light-cone gauge action may be written as

$$\begin{aligned} S &= \frac{1}{2}T \int d\tau \int_0^{2\pi\ell} d\sigma L , \quad T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi} , \\ L &= \dot{x}^* \dot{x} + \left( \dot{z}^M + \frac{ip^+}{z^2} z_N \eta_i (\rho^{MN})^i{}_j \eta^j \right)^2 + ip^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \\ &\quad - \frac{(p^+)^2}{z^2} (\eta^i \eta_i)^2 - \frac{1}{z^4} (x'^* x' + z'^M z'^M) \\ &\quad - 2 \left[ \frac{p^+}{z^3} z^M \eta^i (\rho^M)_{ij} (\theta'^j - \frac{i}{z} \eta^j x') + \frac{p^+}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij} (\theta'_j + \frac{i}{z} \eta_j x'^*) \right] . \end{aligned} \quad (2.6)$$

Here the  $\theta^i = (\theta_i)^\dagger$ ,  $\eta^i = (\eta_i)^\dagger$  ( $i = 1, 2, 3, 4$ ) transform in the fundamental representation of  $SU(4)$  and parametrize the physical fermionic degrees of freedom (the remaining parts of the two ten dimensional Majorana-Weyl spinors in the original GS action).<sup>5</sup> The matrices  $\rho_{ij}^M$  are the off-diagonal blocks of the Dirac matrices in six dimensions in chiral representation and  $\rho^{MN} = \rho^{[M} \rho^{\dagger N]}$  are the  $SO(6)$  generators (see Appendix A).

The action has manifest  $SO(6)$  or  $SU(4)$  symmetry. It is quartic in the  $\eta$ -fermions and quadratic in the  $\theta$ -fermions. As in the flat space light-cone gauge GS action, the factors of  $p^+$  can be absorbed by rescaling the fermions  $\theta_i$  and  $\eta_i$  ( $p^+$  will still appear in the expressions for conserved charges).<sup>6</sup> For generality we introduced the parameter  $\ell$  in the range of  $\sigma$ . For example, if we consider closed string case with the world-sheet topology of a cylinder, before fixing any gauge we can always set  $\ell = 1$  by a coordinate transformation.<sup>7</sup>

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<sup>4</sup>In general, as in flat space case, the knowledge of  $x^-$  is still required to construct the charges of the symmetry algebra and vertex operators. Here, however, we will consider an observable that is determined just by the light-cone gauge action that does not contain  $x^-$ .

<sup>5</sup>Here  $\dagger$  stands for hermitian conjugation on the Grassmann algebra, i.e. fermions are complex.

<sup>6</sup>The above action is related to the one in (1.4),(1.5) in [12] by  $\tau \rightarrow (p^+)^{-1}\tau$ . It is also related to the action in (5.29) in [13] by  $\sigma \rightarrow T^{-1}\sigma$ .

<sup>7</sup>Another choice is to rescale  $\tau$  and  $\sigma$  to set  $x^+ = \tau$  and  $\ell = p^+$  as in the discussion of string interactions in flat space, see also [12].

Before gauge fixing the action is also invariant under  $x^m \rightarrow kx^m$ ,  $z^M \rightarrow kz^M$ . In the gauge-fixed action (2.7) this symmetry becomes  $x \rightarrow kx$ ,  $z^M \rightarrow kz^M$ ,  $\sigma \rightarrow k^{-2}\sigma$ ,  $\ell \rightarrow k^{-2}\ell$ , so we can still set  $\ell = 1$  by such a rescaling. We can also consider the open string case defined on a strip or a half-plane (in the latter case  $\ell = \infty$ ).

In the next two sections we shall consider the string path integral with the two-dimensional euclidean version of the action (2.7), i.e. with  $e^{iS} = e^{-S_E}$ . The euclidean action can be formally obtained from (2.7) by replacing  $\sigma \rightarrow i\sigma$  (and assuming that  $\ell = \infty$ ). Setting  $p^+ = 1$  leads to the action

$$S_E = \frac{1}{2}T \int d\tau \int_0^\infty d\sigma L_E , \quad (2.8)$$

$$\begin{aligned} L_E = & \dot{x}^* \dot{x} + \left( \dot{z}^M + \frac{i}{z^2} z_N \eta_i (\rho^{MN})^i_j \eta^j \right)^2 + i(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \\ & - \frac{1}{z^2} (\eta^i \eta_i)^2 + \frac{1}{z^4} (x'^* x' + z'^M z'^M) \\ & + 2i \left[ \frac{1}{z^3} z^M \eta^i (\rho^M)_{ij} (\theta'^j - \frac{i}{z} \eta^j x') + \frac{1}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij} (\theta'_j + \frac{i}{z} \eta_j x'^*) \right] . \end{aligned} \quad (2.9)$$

Dropping all  $\sigma$ -derivatives in (2.7) gives the light-cone Lagrangian for the  $AdS_5 \times S^5$  superparticle. When quantized [33, 12, 18], it reproduces the spectrum of IIB supergravity on  $AdS_5 \times S^5$ .

The action (2.6),(2.7) has a natural “ground state” – the classical solution

$$z = a = \text{const} , \quad (2.10)$$

$$x^+ = p^+ \tau , \quad x^- = 0 , \quad x^1, x^2, \theta, \eta = 0 . \quad (2.11)$$

This solution – which is the direct counterpart of the point-like limit of the superstring in flat space – describes a massless geodesic parallel to the boundary of the Poincaré patch running at a distance  $a$  from it. It reaches the boundary at spatial infinity ( $x_3 = \infty$ ). The case of  $a = \infty$  corresponds to the massless geodesic passing through the horizon or the center of  $AdS_5$  in global coordinates. In global coordinates this massless geodesic is an arc that reaches the boundary of  $AdS_5$  (and then reflects back).

To describe fluctuations near the solution (2.10) we may set

$$z^M = z_0^M + \tilde{z}^M , \quad z_0^M = e^{\phi_0} u_0^M = a(0, 0, 0, 0, 1) , \quad (2.12)$$

and then the quadratic fluctuation term in (2.7) will take the form (we rescaled the fermions by  $p^+$ )

$$L_2 = \dot{x}^* \dot{x} + \dot{\tilde{z}}^M \dot{\tilde{z}}^M - a^{-4} (x'^* x' + \tilde{z}'^M \tilde{z}'^M)$$

$$+ \left[ i(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i) - 2a^{-2} \eta^i (\rho^6)_{ij} \theta^j + \text{h.c.} \right] , \quad (2.13)$$

where  $\rho^6$  plays the role of the charge conjugation matrix (see [12]). This Lagrangian describes a collection of 8+8 massless excitations, i.e. it is exactly the same action that one finds from flat space GS action when using a similar parametrization of the 16 fermionic coordinates; the only difference is the presence of the “velocity of light” factor  $c = a^{-2}$ .<sup>8</sup>

Since the fluctuation spectrum contains 8 massless 2d bosons and 8 massless 2d fermions the 1-loop string partition function or 1-loop correction to the 2-d energy vanishes [36], in agreement with the fact that the massless geodesic should represent a BPS state. Including quartic interaction terms in (2.13) one may check that the string partition function remains trivial also at the 2-loop order.

Let us comment on the values of conserved charges on the solution (2.10). The expressions for the superconformal charges that correspond to the light-cone gauge action (2.7) were given in [11, 12] and for (2.10) we find that the only non-zero charge densities are<sup>9</sup>

$$P^+ = p^+ , \quad K^+ = -\frac{1}{2}a^2 p^+ . \quad (2.14)$$

Here  $K^+$  represents a component of the special conformal generator  $K^m$  of  $SO(2, 4)$ ; its expectation value thus vanishes only if the geodesic runs directly at the boundary, i.e. if  $a = 0$ .

In general, using the global embedding coordinates ( $\eta^{AB} X_A X_B = -X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = -1$ ) a massless geodesic in  $AdS_5$  is described by (see, e.g., [36]):

$$X_A = N_A + M_A \tau , \quad \eta^{AB} M_A M_B = \eta^{AB} N_A M_B = 0 , \quad \eta^{AB} N_A N_B = -1 \quad (2.15)$$

so that the  $SO(2, 4)$  angular momentum is  $S_{AB} = N_A M_B - N_B M_A$ . In particular, the choice when the motion is along the third spatial direction and  $z = a = 1$  corresponds to  $N_A = (0, 0, 0, 0, 0, 1)$ ,  $M_A = (p, 0, 0, p, 0, 0)$ ; then  $S_{50} = S_{53} = p$ . The relation between  $S_{AB}$  generators and standard basis of conformal group generators on  $R^{1,3}$  is as follows:  $S_{m4} = \frac{1}{2}(K_m - P_m)$ ,  $S_{m5} = \frac{1}{2}(K_m + P_m)$ ,  $S_{54} = D$ ,  $L_{mn} = S_{mn}$  ( $m, n = 0, 1, 2, 3$ ). The global AdS energy is  $E = S_{05} = \frac{1}{2}(K_0 + P_0)$ . In the present case then  $K_m = P_m$  ( $m = 0, 3$ ),  $P^0 = -P^3 = p$  and thus, up to a trivial rescaling, this is the same as in (2.14). Thus the global AdS energy here is same as the Poincaré patch one,  $E = P_0 = P_3$ , but since  $K^m$  is non-zero this does not represent a conformal primary state.

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<sup>8</sup>It can be absorbed by rescaling  $\sigma \rightarrow a^{-2}\sigma$  and then will appear in front of the action together with string tension  $T$  as  $Ta^{-2}$  and also will rescale the length of the cylinder:  $\ell \rightarrow a^2\ell$ .

<sup>9</sup>We use that here  $x^- = 0$ . The charge densities are constant so when integrated they will have a prefactor  $2\pi T\ell = \sqrt{\lambda} \ell$  which we omit here.

### 3 Expansion near null cusp background

Let us now turn to another simple but less trivial solution of the (euclidean) superstring action for which the fluctuation spectrum is massive and the full fluctuation Lagrangian has constant coefficients – the null cusp background [22, 19].<sup>10</sup> Starting with (2.9) one finds<sup>11</sup>

$$z = \sqrt{\frac{\tau}{\sigma}}, \quad x_1 = x_2 = 0. \quad (3.1)$$

In addition we have (we set  $p^+ = 1$ )<sup>12</sup>

$$x^+ = \tau, \quad x^- = -\frac{1}{2\sigma}, \quad x^+ x^- = -\frac{1}{2} z^2. \quad (3.2)$$

This solution is describing a euclidean world surface of an open string ending on the AdS boundary (we assume that  $\tau$  and  $\sigma$  change from 0 to  $\infty$ ). Since  $x^+ x^- = 0$  at  $z = 0$  this surface ends on a null cusp.

Our aim will be to compute the expectation value of the corresponding Wilson loop represented [37, 38] by the euclidean  $AdS_5 \times S^5$  string path integral with the null cusp boundary conditions<sup>13</sup>

$$\langle \mathcal{W}_{\text{cusp}} \rangle = Z_{\text{string}} = \int [dx dz d\theta d\eta] e^{-S_E}. \quad (3.3)$$

The semiclassical computation of this path integral is based on expanding near the solution (3.1). An important feature of this expansion is that it is possible to choose the

<sup>10</sup>As was already mentioned above, the null cusp solution is related [19] by an analytic continuation and a global conformal transformation to the infinite spin limit of the folded string solution [5] which is indeed a “homogeneous” solution [35].

<sup>11</sup>We thank Martin Kruczenski for informing us about this form of the null cusp solution in the AdS light-cone gauge.

<sup>12</sup>Here the analog of the Virasoro constraint gives  $\frac{1}{2}(\dot{x}^+ x'^- + \dot{x}^- x'^+) + \dot{z} z' = 0$ . Let us mention that there is an even simpler solution – a (euclidean) surface ending on a straight line at the boundary,  $x^+ = \tau$ ,  $x^- = b^2 \tau$ ,  $z = \frac{1}{b\sigma}$ ,  $\sigma \in (0, \infty)$ . In this case the string partition function should be trivial, i.e. equal to 1 [38, 12]. While simple at first sight, this solution is however not homogeneous – i.e. the coefficients of the action of small fluctuations are functions of worldsheet coordinates. Thus, carrying out higher order perturbative calculations seems quite involved.

<sup>13</sup>Here it is assumed that  $x^+$  and  $x^-$  are already integrated out using the light-cone gauge conditions. Starting, say, with the Nambu version of superstring action and imposing the orthogonal-gauge conditions on the induced metric (2.4) as well as (2.5) as  $\delta$ -function conditions in the path integral one may then get rid of the integrals over  $x^+$  and  $x^-$ .



fluctuation fields and worldsheet coordinates such that the coefficients of the fluctuation action become constant (i.e. independent of  $\tau, \sigma$ ). Namely, let us define the string coordinate fluctuations by

$$z = \sqrt{\frac{\tau}{\sigma}} \tilde{z}, \quad \tilde{z} = e^{\tilde{\phi}} = 1 + \tilde{\phi} + \dots, \quad z^M = \sqrt{\frac{\tau}{\sigma}} \tilde{z}^M, \quad \tilde{z}^M = e^{\tilde{\phi}} \tilde{u}^M \quad (3.4)$$

$$\tilde{u}^a = \frac{y^a}{1 + \frac{1}{4}y^2}, \quad \tilde{u}^6 = \frac{1 - \frac{1}{4}y^2}{1 + \frac{1}{4}y^2}, \quad y^2 \equiv \sum_{a=1}^5 (y^a)^2, \quad a = 1, \dots, 5, \quad (3.5)$$

$$x = \sqrt{\frac{\tau}{\sigma}} \tilde{x}, \quad \theta = \frac{1}{\sqrt{\sigma}} \tilde{\theta}, \quad \eta = \frac{1}{\sqrt{\sigma}} \tilde{\eta}. \quad (3.6)$$

A further redefinition of the worldsheet coordinates  $(\tau, \sigma) \rightarrow (t, s)$  (we will denote by  $(p_0, p_1)$  the corresponding two-dimensional momenta, i.e.  $(p_0, p_1) = -i(\partial_t, \partial_s)$ )

$$t = \ln \tau, \quad s = \ln \sigma, \quad dt ds = \frac{d\tau d\sigma}{\tau \sigma}, \quad \tau \partial_\tau = \partial_t, \quad \sigma \partial_\sigma = \partial_s. \quad (3.7)$$

leads then to the following euclidean action (2.8), (2.9):

$$S_E = \frac{1}{2} T \int dt \int_{-\infty}^{\infty} ds \mathcal{L}, \quad (3.8)$$

$$\begin{aligned} \mathcal{L} = & \left| \partial_t \tilde{x} + \frac{1}{2} \tilde{x} \right|^2 + \frac{1}{\tilde{z}^4} \left| \partial_s \tilde{x} - \frac{1}{2} \tilde{x} \right|^2 \\ & + \left( \partial_t \tilde{z}^M + \frac{1}{2} \tilde{z}^M + \frac{i}{\tilde{z}^2} \tilde{\eta}_i (\rho^{MN})^i_j \tilde{\eta}^j \tilde{z}_N \right)^2 + \frac{1}{\tilde{z}^4} \left( \partial_s \tilde{z}^M - \frac{1}{2} \tilde{z}^M \right)^2 \\ & + i(\tilde{\theta}^i \partial_t \tilde{\theta}_i + \tilde{\eta}^i \partial_t \tilde{\eta}_i + \tilde{\theta}_i \partial_t \tilde{\theta}^i + \tilde{\eta}_i \partial_t \tilde{\eta}^i) - \frac{1}{\tilde{z}^2} (\tilde{\eta}^2)^2 \\ & + 2i \left[ \frac{1}{\tilde{z}^3} \tilde{\eta}^i (\rho^M)_{ij} \tilde{z}^M (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j - \frac{i}{\tilde{z}} \tilde{\eta}^j (\partial_s x - \frac{1}{2} x)) \right. \\ & \quad \left. + \frac{1}{\tilde{z}^3} \tilde{\eta}_i (\rho_M^\dagger)^{ij} \tilde{z}^M (\partial_s \tilde{\theta}_j - \frac{1}{2} \tilde{\theta}_j + \frac{i}{\tilde{z}} \tilde{\eta}_j (\partial_s x^* - \frac{1}{2} x^*)) \right]. \quad (3.10) \end{aligned}$$

Given that the coefficients in the fluctuation action are constant, we should find for the partition function in (3.3)<sup>14</sup>

$$Z_{\text{string}} = e^{-W}, \quad W = W_0 + W_1 + W_2 + \dots = \frac{1}{2} f(\lambda) V, \quad (3.11)$$

$$V = \frac{1}{4} V_2, \quad V_2 \equiv \int dt \int ds, \quad (3.12)$$

where  $W_0 = S_E$  is the value of the classical action on the solution and  $W_1, W_2, \dots$  are quantum corrections. The cusp anomaly function  $f(\lambda)$  has thus the following inverse string tension expansion

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} \left[ 1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + \frac{a_3}{(\sqrt{\lambda})^3} + \dots \right]. \quad (3.13)$$

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<sup>14</sup>The presence of extra  $\frac{1}{4}$  in the volume factor is due to our choice of unit of scale, see below.

To compute the 1-loop coefficient  $a_1$  let us consider the quadratic part of the fluctuation Lagrangian which identifies the spectrum of excitations<sup>15</sup>

$$\begin{aligned}\mathcal{L}_2 = & (\partial_t \tilde{\phi})^2 + (\partial_s \tilde{\phi})^2 + \tilde{\phi}^2 + |\partial_t \tilde{x}|^2 + |\partial_s \tilde{x}|^2 + \frac{1}{2} |\tilde{x}|^2 + (\partial_t y^a)^2 + (\partial_s y^a)^2 \\ & + 2i (\tilde{\theta}^i \partial_t \tilde{\theta}_i + \tilde{\eta}^i \partial_t \tilde{\eta}_i) + 2i \tilde{\eta}^i (\rho^6)_{ij} (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j) + 2i \tilde{\eta}_i (\rho_6^\dagger)^{ij} (\partial_s \tilde{\theta}_j - \frac{1}{2} \tilde{\theta}_j) .\end{aligned}\quad (3.14)$$

We thus find the same mass spectrum as in conformal gauge [35, 19], up to normalization of the mass scale.<sup>16</sup> The bosonic modes are: one field ( $\tilde{\phi}$ ) with  $m^2 = 1$ ; two fields ( $\tilde{x}, \tilde{x}^*$ ) with  $m^2 = \frac{1}{2}$ ; five fields ( $y^a$ ) with  $m^2 = 0$ .<sup>17</sup> As in (2.13) (or as in flat space), the fermions parametrized by  $\theta^i$  and  $\eta^i$  have an off-diagonal kinetic operator but now with non-zero mass terms<sup>18</sup>

$$\mathcal{L}_F = i\Theta K_F \Theta^T, \quad \Theta = (\theta^i, \theta_i, \eta^i, \eta_i) \equiv (\theta, \theta^\dagger, \eta, \eta^\dagger) \quad (3.15)$$

$$K_F = \begin{pmatrix} 0 & ip_0 \mathbf{1}_4 & -(ip_1 + \frac{1}{2})\rho^6 & 0 \\ ip_0 \mathbf{1}_4 & 0 & 0 & -(ip_1 + \frac{1}{2})\rho_6^\dagger \\ +(ip_1 - \frac{1}{2})\rho^6 & 0 & 0 & ip_0 \mathbf{1}_4 \\ 0 & +(ip_1 - \frac{1}{2})\rho_6^\dagger & ip_0 \mathbf{1}_4 & 0 \end{pmatrix}. \quad (3.16)$$

The matrices  $\rho^6$  (carrying lower indices) and  $\rho_6^\dagger$  (carrying upper indices) are related as in Appendix A. The determinant of the fermionic kinetic operator is  $\det K_F = (p^2 + \frac{1}{4})^8$  implying that all 8 physical fermionic degrees of freedom have  $m^2 = \frac{1}{4}$ . The equality of masses of all the fermionic modes is required by the  $SO(6)$  symmetry of the null cusp background [31].

Having the same mass spectrum implies the same (UV finite) result for the 1-loop

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<sup>15</sup>Here we used that  $\theta^i = \theta_i^\dagger$ ,  $\eta^i = \eta_i^\dagger$  and ignored a total derivative term.

<sup>16</sup>Here the classical solution (3.1) is  $z = e^{\frac{1}{2}(t-s)}$  which differs by a rescaling of  $s$  and  $t$  from the form used in [19]. The mass scales in the light-cone and the conformal gauges are related as  $m_{\text{l.c.}}^2 = \frac{1}{4} m_{\text{conf.}}^2$ .

<sup>17</sup>It is interesting to note that the analogs of the first three modes in the closed string picture (i.e. for fluctuations near the long folded spinning string [5] in  $AdS_3$  part of  $AdS_5$ ) are the angular  $AdS_3$  mode “transverse” to the profile of the string and the two  $AdS_5$  modes “transverse” to the  $AdS_3$  subspace of the solution [6, 35].

<sup>18</sup>Whenever indices on fermions are not written explicitly we will implicitly assume that  $\theta$  and  $\eta$  carry upper indices while  $\theta^\dagger$  and  $\eta^\dagger$  carry lower indices.

partition function as found in conformal gauge [6, 35, 19, 20]:<sup>19</sup>

$$\begin{aligned} W_1 = -\ln Z_1 &= \frac{1}{2}V_2 \int \frac{d^2p}{(2\pi)^2} \left[ \ln(p^2 + 1) + 2\ln(p^2 + \tfrac{1}{2}) + 5\ln p^2 - 8\ln(p^2 + \tfrac{1}{4}) \right] \\ &= -\frac{3\ln 2}{8\pi}V_2, \end{aligned} \quad (3.17)$$

i.e. we get  $a_1 = -3\ln 2$  in (3.13).

In the next section we shall extend this computation to the 2-loop level and show that, as in the conformal gauge [20, 21], the 2-loop coefficient in (3.13) is minus the Catalan's constant

$$a_2 = -K, \quad K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.9159\dots \quad (3.18)$$

## 4 Cusp anomaly at 2-loops

An important feature of the light-cone gauge action expanded near the cusp solution is that the bosonic propagator is diagonal. This is a useful simplification for higher loop calculations as we shall now demonstrate by the explicit computation of the 2-loop coefficient (3.18) in the cusp anomaly. Finding  $a_2$  amounts to computing all connected vacuum Feynman diagrams in the background of the null cusp (3.1). We will thus need to expand the light-cone gauge Lagrangian (3.9) to the quartic order.<sup>20</sup>

### 4.1 One-particle irreducible contributions

We begin by analyzing the one-particle irreducible contributions to the partition function. At 2-loops they correspond to the sunset and double-bubble diagrams, see fig. 1. The

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<sup>19</sup>Note that with the choice of normalization we use here the light-cone and the conformal gauge volume factors are related by  $V_2 = 4V_2^{\text{conf.}}$ .

<sup>20</sup>Let us note that in [20] the coefficient  $a_2$  was calculated in the conformal gauge by using a T-dual version of the  $AdS_5 \times S^5$  action in the Poincaré patch coordinates. This approach is convenient because the T-dual action is only quadratic in the fermions [39]. To get such an action one must fix the  $\kappa$ -symmetry by choosing the so called S-gauge [11]. One may wonder whether it is possible to combine the virtues of the bosonic light cone gauge  $x^+ = \tau$  with the simplicity of the T-dual Green-Schwarz action. It turns out that the choice of the S-gauge is not compatible with the bosonic light-cone gauge. Indeed, the equation of motion for  $x^-$  would be  $0 = d * dx^+ + (\delta_{IJ} d * + s_{IJ} d) \bar{\vartheta}^I \Gamma^+ d\vartheta^J = (\delta_{IJ} d * + s_{IJ} d) \bar{\vartheta}^I \Gamma^+ d\vartheta^J$ , which is in contradiction with the fact that in the S-gauge  $(\delta_{IJ} d * + s_{IJ} d) \bar{\vartheta}^I \Gamma^+ d\vartheta^J \neq 0$ . As usual,  $s_{IJ} = \text{diag}(1, -1)$ .

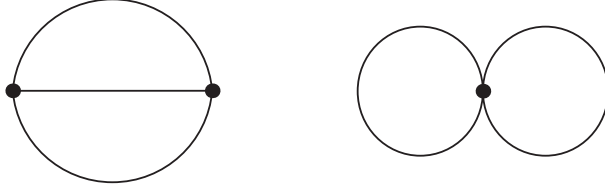


Figure 1: The 2-loop 1PI topologies. The propagators represent either bosonic or fermionic fluctuations.

various contributions to the 2-loop part of  $W = -\ln Z_{\text{string}}$  are obtained from

$$W_2 = \langle S_{\text{int}} \rangle - \frac{1}{2} \langle S_{\text{int}}^2 \rangle_c + \dots, \quad (4.1)$$

where  $S_{\text{int}}$  is the interacting part of the action (3.8),(3.9) containing cubic and quartic terms. As usual, the Wick contractions are made by inserting the appropriate propagators, and the subscript  $c$  indicates that only connected diagrams are to be included. At the 2-loop level, the first term in (4.1) gives the double-bubble diagram while the second term gives the sunset diagram as well as the connected graph with two tadpoles which will be discussed in the next subsection.

For the bosonic sunset diagrams we need the following cubic terms from the action,<sup>21</sup>

$$\begin{aligned} S_{\tilde{\phi}\tilde{x}\tilde{x}^*}^{(3)} &= -2 \int dt ds \tilde{\phi} |\partial_s \tilde{x} - \frac{1}{2} \tilde{x}|^2 \\ S_{\tilde{\phi}^3}^{(3)} &= \int dt ds \tilde{\phi} [(\partial_t \tilde{\phi})^2 - (\partial_s \tilde{\phi})^2] \\ S_{\tilde{\phi}y^2}^{(3)} &= \int dt ds \tilde{\phi} [(\partial_t y^a)^2 - (\partial_s y^a)^2]. \end{aligned} \quad (4.2)$$

The fact that the bosonic propagator is diagonal implies that the sunset graph is simply given by

$$W_{2 \text{ bos.sunset}} = -\frac{1}{2} \langle S_{\tilde{\phi}\tilde{x}\tilde{x}}^{(3)} S_{\tilde{\phi}\tilde{x}\tilde{x}}^{(3)} + S_{\tilde{\phi}^3}^{(3)} S_{\tilde{\phi}^3}^{(3)} + S_{\tilde{\phi}y^2}^{(3)} S_{\tilde{\phi}y^2}^{(3)} \rangle_{\text{1PI}}. \quad (4.3)$$

All the terms in the above expression can be readily computed. For instance, the first term yields, in momentum space,

$$\begin{aligned} &-2 \int d^2 p d^2 q d^2 r \delta^{(2)}(p+q+r) G_{\tilde{\phi}\tilde{\phi}}(p) (q_1^2 + \frac{1}{4}) G_{\tilde{x}\tilde{x}}(q) (r_1^2 + \frac{1}{4}) G_{\tilde{x}\tilde{x}}(r) \\ &= -\frac{1}{2} \int d^2 p d^2 q d^2 r \delta^{(2)}(p+q+r) \frac{(1+4q_1^2)(1+4r_1^2)}{(p^2+1)(q^2+\frac{1}{2})(r^2+\frac{1}{2})}, \end{aligned} \quad (4.4)$$

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<sup>21</sup>Here we temporarily set the string tension  $T$  to one. In the following, we will sometimes ignore also the obvious 2-d volume factor  $V_2$ . The dependence on  $T$  and  $V_2$  is easily reinserted at the end of the calculation.

where we have used the propagators for the bosonic fluctuations in (3.14)

$$G_{\tilde{x}\tilde{x}^*}(p) = \frac{2}{p^2 + \frac{1}{2}}, \quad G_{\tilde{\phi}\tilde{\phi}}(p) = \frac{1}{p^2 + 1}, \quad G_{y^a y^b}(p) = \frac{\delta^{ab}}{p^2}. \quad (4.5)$$

To evaluate the momentum integrals, we employ the same regularization scheme used in [20, 21]. Manipulation of tensor structures in the numerators are performed in  $d = 2$ , and the resulting scalar integrals are computed in an analytic (e.g., dimensional) regularization scheme in which power divergent contributions are set to zero.<sup>22</sup> Namely, we will set

$$\int \frac{d^2 p}{(2\pi)^2} (p^2)^n = 0, \quad n \geq 0. \quad (4.6)$$

Introducing the notations [20, 21]

$$I[m^2] = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{(p^2 + m^2)} \quad (4.7)$$

$$I[m_1^2, m_2^2, m_3^2] = \int \frac{d^2 p d^2 q d^2 r}{(2\pi)^4} \frac{\delta^2(p + q + r)}{(p^2 + m_1^2)(q^2 + m_2^2)(r^2 + m_3^2)}, \quad (4.8)$$

we obtain for the above integral

$$\frac{1}{2} \int d^2 p d^2 q d^2 r \delta^{(2)}(p + q + r) \frac{(1 + 4q_1^2)(1 + 4r_1^2)}{(p^2 + 1)(q^2 + \frac{1}{2})(r^2 + \frac{1}{2})} = -\frac{1}{4} I[1, \frac{1}{2}, \frac{1}{2}]. \quad (4.9)$$

This integral is proportional to the Catalan's constant in (3.18) since in general

$$I[2m^2, m^2, m^2] = \frac{K}{8\pi^2 m^2}. \quad (4.10)$$

The computation of the remaining contributions is analogous.

The second term in (4.3) gives a result proportional to  $I[1]^2$ , while the last term turns out to vanish. When everything is put together, we obtain the following simple answer for the bosonic sunset diagram

$$W_{2 \text{ bos.sunset}} = \frac{1}{4} I[1, \frac{1}{2}, \frac{1}{2}] + \frac{1}{2} I[1]^2. \quad (4.11)$$

Note that for non-zero masses the integral  $I[m_1^2, m_2^2, m_3^2]$  is finite, while  $I[m^2]$  is logarithmically UV divergent. When any of the masses vanishes, both types of integrals exhibit IR singularities.

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<sup>22</sup>The direct cancellation of these divergences amounts to carefully accounting for the contribution of the path integral measure and was shown to occur in the conformal gauge [21].

Let us now consider the bosonic double-bubble diagram. This is given by

$$W_{2 \text{ bos.bubble}} = \langle S^{(4)} \rangle, \quad (4.12)$$

where  $S^{(4)}$  includes the following quartic vertices,

$$S_{\tilde{\phi}^2 \tilde{x} \tilde{x}^*}^{(4)} = 4 \int dt ds \tilde{\phi}^2 |\partial_s \tilde{x} - \frac{1}{2} \tilde{x}|^2 \quad (4.13)$$

$$S_{\tilde{\phi}^4}^{(4)} = \int dt ds \tilde{\phi}^2 [(\partial_t \tilde{\phi})^2 + (\partial_s \tilde{\phi})^2 + \frac{1}{6} \tilde{\phi}^2] \quad (4.14)$$

$$S_{\tilde{\phi}^2 y^2}^{(4)} = \int dt ds \tilde{\phi}^2 [(\partial_t y^a)^2 + (\partial_s y^a)^2] \quad (4.15)$$

$$S_{y^4}^{(4)} = -\frac{1}{4} \int dt ds y^a y^a (\partial_t y^b \partial_t y^b + \partial_s y^b \partial_s y^b). \quad (4.16)$$

It turns out that the only non-vanishing contribution comes from the  $\tilde{\phi}^4$ -interaction, and the final result is

$$W_{2 \text{ bos.bubble}} = -\frac{1}{2} \text{I}[1]^2. \quad (4.17)$$

Next, let us consider the vertices coming from the fermionic part of the Green-Schwarz action. For the sunset diagram we need the following cubic interactions,

$$\begin{aligned} S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} &= -2i \int dt ds \tilde{\eta}^i (\rho^6)_{ij} (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j) \tilde{\phi} - \text{h.c.} \\ S_{y \tilde{\eta} \tilde{\theta}}^{(3)} &= +i \int dt ds \tilde{\eta}^i (\rho^a)_{ij} (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j) y^a - \text{h.c.} \\ S_{\tilde{x} \tilde{\eta} \tilde{\eta}}^{(3)} &= + \int dt ds \tilde{\eta}^i (\rho^6)_{ij} \tilde{\eta}^j (\partial_s \tilde{x} - \frac{1}{2} \tilde{x}) - \text{h.c.} \\ S_{y \tilde{\eta} \tilde{\eta}}^{(3)} &= +i \int dt ds \tilde{\eta}_i (\rho^{a6})^i_j \tilde{\eta}^j \partial_t y^a. \end{aligned} \quad (4.18)$$

The fact that the bosonic propagator is diagonal leads to a dramatic reduction of the number of possible terms. The fermionic contribution to the sunset diagram is

$$W_{2 \text{ ferm.sunset}} = -\frac{1}{2} \langle S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} + S_{\tilde{x} \tilde{\eta} \tilde{\eta}}^{(3)} S_{\tilde{x} \tilde{\eta} \tilde{\eta}}^{(3)} + S_{y \tilde{\eta} \tilde{\eta}}^{(3)} S_{y \tilde{\eta} \tilde{\eta}}^{(3)} + S_{y \tilde{\eta} \tilde{\theta}}^{(3)} S_{y \tilde{\eta} \tilde{\theta}}^{(3)} + 2 S_{y \tilde{\eta} \tilde{\eta}}^{(3)} S_{y \tilde{\eta} \tilde{\theta}}^{(3)} \rangle_{\text{1PI}}. \quad (4.19)$$

As an example of a typical calculation let us detail the analysis of  $\langle S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} \rangle_{\text{1PI}}$ ; Wick contractions yield the following expression

$$-\frac{1}{2} \langle S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} S_{\tilde{\phi} \tilde{\eta} \tilde{\theta}}^{(3)} \rangle_{\text{1PI}} = -\frac{1}{2} (2i)^2 G_{\tilde{\phi} \tilde{\phi}}(r) \left[ (ip_1 - \frac{1}{2})(iq_1 + \frac{1}{2})A - (q_1^2 + \frac{1}{4})B \right] \quad (4.20)$$

where

$$\begin{aligned} A &= \text{Tr} \left[ \rho_6^\dagger G_{\tilde{\theta}^\dagger \tilde{\eta}^\dagger}(p) \rho_6^\dagger G_{\tilde{\theta}^\dagger \tilde{\eta}^\dagger}(-q) + \rho^6 G_{\tilde{\theta} \tilde{\eta}}(p) \rho^6 G_{\tilde{\theta} \tilde{\eta}}(-q) \right] \\ B &= \text{Tr} \left[ \rho^6 G_{\tilde{\eta} \tilde{\eta}^\dagger}(p) \rho_6^\dagger G_{\tilde{\theta}^\dagger \tilde{\theta}}(-q) + \rho_6^\dagger G_{\tilde{\eta}^\dagger \tilde{\eta}}(p) \rho^6 G_{\tilde{\theta} \tilde{\theta}^\dagger}(-q) \right]. \end{aligned} \quad (4.21)$$

The fermion propagators appearing in this expression are (proportional to) the relevant entries of the inverse of the fermionic kinetic operator (3.16), and are given by

$$\begin{aligned} G_{\tilde{\theta}^i \tilde{\eta}^j}(p) &= -\frac{p_1 - \frac{i}{2}}{p^2 + \frac{1}{4}} \rho_6^\dagger, & G_{\tilde{\theta}^i \tilde{\eta}^j}(p) &= -\frac{p_1 - \frac{i}{2}}{p^2 + \frac{1}{4}} \rho^6, \\ G_{\tilde{\theta}^i \tilde{\theta}^j}(p) &= G_{\tilde{\eta}^i \tilde{\eta}^j}(p) = -\frac{p_0}{p^2 + \frac{1}{4}} \mathbf{1}_4. \end{aligned} \quad (4.22)$$

After collecting all contributions in (4.19) and reducing them to scalar integrals, the final result for the fermionic sunset diagram turns out to be

$$W_{2 \text{ ferm.sunset}} = -\frac{1}{4} I[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}] + 2 I[\frac{1}{4}]^2 + 2 I[\frac{1}{4}] I[1] - \frac{5}{2} I[\frac{1}{4}] I[0]. \quad (4.23)$$

Finally, we have to include the fermionic contributions to the double-bubble topology. It is easy to see that the diagram with two fermion bubbles, which, in principle, arises due to the  $\tilde{\eta}^4$  interaction, vanishes, and so do all diagrams with an  $\tilde{\eta}\tilde{\eta}$ -loop. Then the only non-trivial contributions come from the following boson-fermion 4-vertices

$$S_{yy\tilde{\eta}\tilde{\theta}}^{(4)} = -\frac{i}{2} \int dt ds y^a y^a \tilde{\eta}^i (\rho^6)_{ij} (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j) - \text{h.c.} \quad (4.24)$$

$$S_{\tilde{\phi}\tilde{\phi}\tilde{\eta}\tilde{\theta}}^{(4)} = +2i \int dt ds \tilde{\phi}^2 \tilde{\eta}^i (\rho^6)_{ij} (\partial_s \tilde{\theta}^j - \frac{1}{2} \tilde{\theta}^j) - \text{h.c.} \quad (4.25)$$

After reduction to scalar integrals we obtain the following result

$$W_{2 \text{ ferm.bubble}} = -2 I[\frac{1}{4}] I[1] + \frac{5}{2} I[\frac{1}{4}] I[0]. \quad (4.26)$$

Thus, the bosonic and fermionic one-particle irreducible contributions, (4.11), (4.12), (4.23) and (4.26), sum up to a divergent 2-loop correction to the partition function. This UV divergence is of  $\log^2$ -type and thus should be canceled by additional non-1PI connected diagram contributions to restore the expected 2-loop finiteness of the superstring theory. This is indeed what happens as we shall see below.

## 4.2 Additional connected graph contribution

So far we have considered only the one-particle irreducible contributions; however,  $\ln Z_{\text{string}}$  receives contributions from all connected graphs. In particular, at two loops we might have non-vanishing tadpole diagrams of the topology given in fig. 2. We will see that such tadpole diagrams play an important role for reproducing the 2-loop result for the cusp anomaly found previously in the conformal gauge.

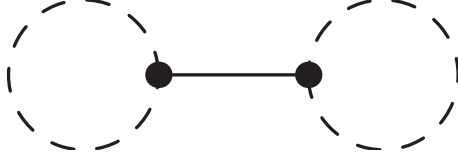


Figure 2: The 2-loop tadpole topology. The only non-vanishing contribution corresponds to fermionic  $\tilde{\theta}\tilde{\eta}$  bubbles, represented by dashed lines, connected by the  $\tilde{\phi}$  propagator.

The (spontaneously broken) symmetries of the theory forbid single-fermion terms from appearing in the effective action. Thus, only the bosonic fields may exhibit nontrivial 1-point functions. Let us begin by discussing the contributions of bosonic loops. Given the structure of the bosonic 3-vertices (4.2) and the fact that the bosonic propagator is diagonal, it is easy to see that all 2-loop graphs with bosonic tadpoles must have  $\tilde{\phi}$  as the inner leg, which is at zero momentum by momentum conservation. Moreover, diagrams with a  $\tilde{\phi}^2$  or a  $y^2$  bubble vanish identically by  $t \leftrightarrow s$  symmetry due to the form of the corresponding vertices. Thus, the only potentially non-trivial terms come from two  $\tilde{x}\tilde{x}^*$  loops and the  $S_{\tilde{\phi}\tilde{x}\tilde{x}^*}^{(3)}$  interaction. Each bubble contributes a factor

$$-4 \int \frac{d^2 p}{(2\pi)^2} (p_1^2 + \frac{1}{4}) G_{\tilde{x}\tilde{x}^*}(p) = -8 \int \frac{d^2 p}{(2\pi)^2} \frac{p_1^2 + \frac{1}{4}}{p^2 + \frac{1}{2}} = -4 \int \frac{d^2 p}{(2\pi)^2}, \quad (4.27)$$

which is zero in our regularization scheme, as explained above. Therefore, we conclude that all bosonic tadpoles vanish.

Since all bosonic non-1PI diagrams vanish identically, the total bosonic contribution comes from summing up the expressions (4.11) and (4.12), i.e. is given by

$$W_{2 \text{ bos}} = \frac{1}{4} \frac{2\pi}{\sqrt{\lambda}} V_2 \text{I}[1, \frac{1}{2}, \frac{1}{2}], \quad (4.28)$$

where we have reinstated the explicit dependence on the inverse of the string tension  $T^{-1} = \frac{2\pi}{\sqrt{\lambda}}$  and the overall two-dimensional volume factor  $V_2$ .

Let us now turn to the analysis of the fermionic tadpoles. Since the  $\tilde{\eta}\tilde{\eta}^*$  two-point function  $G_{\tilde{\eta}\tilde{\eta}^*}(p) = -\frac{p_0}{p^2 + \frac{1}{4}} \mathbf{1}_4$  is parity-odd, the bubble integral containing only this propagator vanishes identically. Thus, the potentially non-trivial tadpoles may come only from the vertices in the first two lines of (4.18). The  $S_{y\tilde{\eta}\tilde{\theta}}^{(3)}$  vertex, however, leads to a vanishing result since each bubble is proportional to  $\text{Tr} [\rho^a G_{\tilde{\eta}\tilde{\theta}}(p)] \propto \text{Tr} \rho^{a6} = 0$ . The remaining



fermionic tadpole with a  $\tilde{\phi}$  internal leg is, on the other hand, non-trivial and gives

$$-\frac{1}{2}\langle S_{\tilde{\phi}\tilde{\eta}\tilde{\theta}}^{(3)}S_{\tilde{\phi}\tilde{\eta}\tilde{\theta}}^{(3)}\rangle_{\text{non-1PI}} = -\frac{1}{2}(2i)^2 G_{\tilde{\phi}\tilde{\phi}}(0) \left( \int \frac{d^2p}{(2\pi)^2} (ip_1 - \frac{1}{2}) \text{Tr}[-(\rho_6^\dagger)G_{\tilde{\theta}^\dagger\tilde{\eta}^\dagger}(p) - (\rho^6)G_{\tilde{\theta}\tilde{\eta}}(p)] \right)^2, \quad (4.29)$$

which after reduction to scalar integrals yields

$$W_{2 \text{ ferm.tadpole}} = -2 \text{I}[\frac{1}{4}]^2. \quad (4.30)$$

This  $\log^2$  divergent term is precisely what we need to cancel a similar divergent term in  $W_{2 \text{ ferm.sunset}}$ , see (4.23). The presence of the tadpole is therefore necessary to guarantee a finite answer for the cusp anomaly.

Let us mention that it is the “vacuum-vacuum” transition amplitude or the background partition function (3.3) that is a physical observable that should be UV finite. As for the effective action  $\Gamma$  given by the sum of 1PI graphs evaluated in a non-trivial background, it is, in general, UV finite only after a field renormalization. The presence of the tadpole for  $\tilde{\phi}$  means that here one would need such a renormalization to make  $\Gamma$  finite. We do not need to worry about this renormalization if our interest is to compute the full partition function in (3.3).

Combining all the partial results we find the answer for  $W$  at 2-loops in light-cone gauge,

$$\begin{aligned} W_2 &= W_{2 \text{ bos.sunset}} + W_{2 \text{ bos.bubble}} + W_{2 \text{ ferm.sunset}} + W_{2 \text{ ferm.bubble}} + W_{2 \text{ ferm.tadpole}} \\ &= \frac{2\pi}{\sqrt{\lambda}} V_2 \left( \frac{1}{4} \text{I}[1, \frac{1}{2}, \frac{1}{2}] - \frac{1}{4} \text{I}[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}] \right) \\ &= -\frac{1}{4} \frac{2\pi}{\sqrt{\lambda}} V_2 \text{I}[1, \frac{1}{2}, \frac{1}{2}] = -\frac{\text{K}}{8\pi\sqrt{\lambda}} V_2. \end{aligned} \quad (4.31)$$

The result is manifestly finite and reproduces the value of  $a_2$  in (3.18). We observe that, as in the conformal gauge calculation of [20, 21], the net effect of the fermions is to change the sign of the bosonic result for  $W_2$ .

We conclude that the AdS light-cone gauge result is thus in perfect agreement with the string theory computation in the conformal gauge [20, 21] and with the strong-coupling prediction of the Bethe ansatz [25].

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## Appendix A: Notation

We mostly follow the notation of [11] but define  $x^\pm$  and  $x, x^*$  without  $\frac{1}{\sqrt{2}}$  factors. Four-dimensional indices (along the AdS boundary) are  $a, b = 0, 1, 2, 3$ ;  $SO(6)$  indices are  $M, N = 1, \dots, 6$ ;  $SU(4)$  indices are  $i, j = 1, 2, 3, 4$ . For the fermionic variables we have  $\theta_i^\dagger = \theta^i$ ,  $\eta_i^\dagger = \eta^i$ ,  $\theta^2 \equiv \theta^i \theta_i$ ,  $\eta^2 \equiv \eta^i \eta_i$ .

The matrices  $\rho^M$  are off-diagonal blocks of the six-dimensional Dirac matrices in chiral representation:

$$\rho_{ij}^M = -\rho_{ji}^M, \quad (\rho^M)^{il} \rho_{lj}^N + (\rho^N)^{il} \rho_{lj}^M = 2\delta^{MN} \delta_j^i, \quad (\rho^M)^{ij} \equiv -(\rho_{ij}^M)^* \quad (\text{A.1})$$

$$\rho^{MNi}{}_j = \frac{1}{2} [(\rho^M)^{il} \rho_{lj}^N - (\rho^N)^{il} \rho_{lj}^M]. \quad (\text{A.2})$$

One can choose the following explicit representation for the  $\rho_{ij}^M$  matrices

$$\begin{aligned} \rho_{ij}^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & \rho_{ij}^2 &= \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & \rho_{ij}^3 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \rho_{ij}^4 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \rho_{ij}^5 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \rho_{ij}^6 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

As usual, their explicit form is not needed to carry out the calculations described in the text. We found it convenient however to use at times the representation described here.

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